

Fixed Effects and Random Effects Models

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1 Fixed Effects Model

We consider a panel model with individual and time fixed effects

$$y_{it} = \pi + \mu_i + \nu_t + x'_{it}\beta + \varepsilon_{it}$$

for $i = 1, \dots, a$ and $t = 1, \dots, b$ where $x_{it} \in \mathbb{R}^m$. In matrix form, we have

$$Y = (\iota_a \iota'_b) \pi + \mu \iota'_b + \iota_a \nu' + X \cdot \beta + E, \quad (1)$$

where $\mu = (\mu_1, \dots, \mu_a)' \in \mathbb{R}^a$, $\nu = (\nu_1, \dots, \nu_b)' \in \mathbb{R}^b$, $Y = (y_{it})_{i,t=1}^{a,b} \in \mathbb{R}^{a \times b}$, $E = (\varepsilon_{it})_{i,t=1}^{a,b} \in \mathbb{R}^{a \times b}$. We define $X \cdot \beta = (x'_{it}\beta)_{i,t=1}^{a,b} \in \mathbb{R}^{a \times b}$.¹ Recall that for row-major vectorization, we have

$$\text{vec}(uv') = (u \otimes v).$$

Thus, vectorizing the matrix form (1), we have

$$\begin{aligned} y &= \iota_n \pi + (\mu \otimes \iota_b) + (\iota_a \otimes \nu) + X\beta + \varepsilon \\ &= \iota_n \pi + (I_a \otimes \iota_b) \mu + (\iota_a \otimes I_b) \nu + X\beta + \varepsilon, \end{aligned} \quad (2)$$

where $n = ab$, $y = \text{vec}(Y) \in \mathbb{R}^n$, $\varepsilon = \text{vec}(E) \in \mathbb{R}^n$ and X is $n \times m$ dimensional matrix whose j -th row is given by x_{it} with $j = (i-1)a + t$.

1.1 Identification

Note that the matrix

$$\begin{pmatrix} \iota_n & (I_a \otimes \iota_b) & (\iota_a \otimes I_b) \end{pmatrix} \in \mathbb{R}^{n \times (a+b+1)}$$

has rank $a + b - 1$. Without loss of generality, we assume

$$\iota'_a \mu = \iota'_b \nu = 0 \quad (3)$$

¹In tensor decomposition literature, $X \cdot \beta$ is defined as mode-3 product applied to the tensor $X \in \mathbb{R}^{a \times b \times m}$, i.e., $X \cdot \beta := X \times_3 \beta' \in \mathbb{R}^{a \times b}$.

for identification. Let $H_a \in \mathbb{R}^{a \times (a-1)}$ and $H_b \in \mathbb{R}^{b \times (b-1)}$ be orthonormal eigenvectors of M_{ι_a} and M_{ι_b} given by

$$M_{\iota_a} = I_a - \frac{\iota_a \iota_a'}{a} = H_a H_a', \quad M_{\iota_b} = I_b - \frac{\iota_b \iota_b'}{b} = H_b H_b'.$$

Then, the identifying restrictions in (3) imply

$$\begin{aligned} \mu &= \left(I_a - \frac{\iota_a \iota_a'}{a} \right) \mu = H_a H_a' \mu = H_a \mu^* \\ \nu &= \left(I_b - \frac{\iota_b \iota_b'}{b} \right) \nu = H_b H_b' \nu = H_b \nu^*, \end{aligned}$$

where $\mu^* := H_a' \mu \in \mathbb{R}^{a-1}$ and $\nu^* := H_b' \nu \in \mathbb{R}^{b-1}$. Thus, the model (2) can be written as

$$y = \iota_n \pi + (H_a \otimes \iota_b) \mu^* + (\iota_a \otimes H_b) \nu^* + X\beta + \varepsilon.$$

1.2 Estimation

Note that the generated regressors

$$\begin{pmatrix} \iota_n & (H_a \otimes \iota_b) & (\iota_a \otimes H_b) \end{pmatrix} \in \mathbb{R}^{n \times (a+b-1)}$$

now have full rank. Furthermore, ι_n , $(H_a \otimes \iota_b)$, and $(\iota_a \otimes H_b)$ are pairwise orthogonal, e.g.,

$$\iota_n' (H_a \otimes \iota_b) = (\iota_a \otimes \iota_b)' (H_a \otimes \iota_b) = (\iota_a' H_a \otimes \iota_b' \iota_b) = 0.$$

Let P be joint projection on ι_n , $(H_a \otimes \iota_b)$, and $(\iota_a \otimes H_b)$. Due to orthogonality, we have

$$P = P_{\iota_n} + P_{H_a \otimes \iota_b} + P_{\iota_a \otimes H_b}.$$

Note that $P_{A \otimes B} = P_A \otimes P_B$ so that

$$\begin{aligned} P_{\iota_n} &= P_{\iota_a \otimes \iota_b} = P_{\iota_a} \otimes P_{\iota_b} \\ P_{H_a \otimes \iota_b} &= P_{H_a} \otimes P_{\iota_b} = M_{\iota_a} \otimes P_{\iota_b} \\ P_{\iota_a \otimes H_b} &= P_{\iota_a} \otimes P_{H_b} = P_{\iota_a} \otimes M_{\iota_b}. \end{aligned}$$

More explicitly, we have

$$P_{\iota_a} \otimes P_{\iota_b} = \frac{\iota_n \iota_n'}{n}, \quad M_{\iota_a} \otimes P_{\iota_b} = \left(I_a - \frac{\iota_a \iota_a'}{a} \right) \otimes \frac{\iota_b \iota_b'}{b}, \quad P_{\iota_a} \otimes M_{\iota_b} = \frac{\iota_a \iota_a'}{a} \otimes \left(I_b - \frac{\iota_b \iota_b'}{b} \right).$$

Using FWL theorem, we have

$$\hat{\beta} = (X'(I_n - P)X)^{-1} X'(I_n - P)y. \quad (4)$$

Let $(\bar{y}_i, \bar{x}_i)$, $(\bar{y}_t, \bar{x}_t)$, and $(\bar{y}, \bar{x})$ be the cross-sectional, time, and total averages of (y_{it}, x_{it}) respectively. We may see that $\hat{\beta}$ in (4) is the OLS estimate in the regression

$$y_{it} - \bar{y}_i - \bar{y}_t + \bar{y} = (x_{it} - \bar{x}_i - \bar{x}_t + \bar{x})' \beta + \varepsilon_{it}$$

To see this, note that

$$(I_n - P)y = (I_n - (P_{\iota_a} \otimes P_{\iota_b}) - (M_{\iota_a} \otimes P_{\iota_b}) - (P_{\iota_a} \otimes M_{\iota_b}))y.$$

Here, $(P_{\iota_a} \otimes P_{\iota_b})y = \bar{y}\iota_n$. Also,

$$\begin{aligned} (M_{\iota_a} \otimes P_{\iota_b})y &= \left(\left(I_a - \frac{\iota_a \iota_a'}{a} \right) \otimes \frac{\iota_b \iota_b'}{b} \right) y \\ &= \left(\left(I_a - \frac{\iota_a \iota_a'}{a} \right) \otimes \frac{\iota_b \iota_b'}{b} \right) \text{vec}(Y) \\ &= \text{vec} \left(\left(I_a - \frac{\iota_a \iota_a'}{a} \right) Y \left(\frac{\iota_b \iota_b'}{b} \right) \right), \end{aligned}$$

so that $(M_{\iota_a} \otimes P_{\iota_b})y$ averages over time and demeanes over cross-section. To be explicit, let

$$Y = \begin{pmatrix} y_1' \\ \vdots \\ y_a' \end{pmatrix} \in \mathbb{R}^{a \times b}. \text{ Then,}$$

$$Y \left(\frac{\iota_b \iota_b'}{b} \right) = \begin{pmatrix} y_1' \\ \vdots \\ y_a' \end{pmatrix} \left(\frac{\iota_b \iota_b'}{b} \right) = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_a \end{pmatrix} \iota_b'$$

where $(\bar{y}_1, \dots, \bar{y}_a)$ are time averages. Furthermore,

$$\left(I_a - \frac{\iota_a \iota_a'}{a} \right) Y \left(\frac{\iota_b \iota_b'}{b} \right) = \left(I_a - \frac{\iota_a \iota_a'}{a} \right) \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_a \end{pmatrix} \iota_b' = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_a \end{pmatrix} \iota_b' - \bar{y}(\iota_a \iota_b')$$

so that

$$(M_{\iota_a} \otimes P_{\iota_b})y = \text{vec} \left(\begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_a \end{pmatrix} \iota_b' - \bar{y}(\iota_a \iota_b') \right) = \left(\begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_a \end{pmatrix} \otimes \iota_b \right) - \iota_n \bar{y}. \quad (5)$$

Similarly,

$$(P_{\iota_a} \otimes M_{\iota_b})y = \left(\iota_a \otimes \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_b \end{pmatrix} \right) - \iota_n \bar{y}, \quad (6)$$

where $(\bar{y}_1, \dots, \bar{y}_b)$ in (6) are now cross-sectional averages. Thus,

$$(I - P)y = y - \left(\begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_a \end{pmatrix} \otimes \iota_b \right) - \left(\iota_a \otimes \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_b \end{pmatrix} \right) + \iota_n \bar{y}$$

Similar argument can be made for the term $(I_n - P)X$ after we write $X = (x_1, \dots, x_m) \in \mathbb{R}^{n \times m}$ where $(x_i)_{i=1}^m$ denotes each columns of X . In other words, we have $(P_{\iota_a} \otimes P_{\iota_b})X = (P_{\iota_a} \otimes P_{\iota_b})(x_1, \dots, x_m) = ((P_{\iota_a} \otimes P_{\iota_b})x_1, \dots, (P_{\iota_a} \otimes P_{\iota_b})x_m) = \iota_n \bar{x}'$. Also,

$$\begin{aligned} (M_{\iota_a} \otimes P_{\iota_b})X &= (M_{\iota_a} \otimes P_{\iota_b})(x_1, \dots, x_m) = ((M_{\iota_a} \otimes P_{\iota_b})x_1, \dots, (M_{\iota_a} \otimes P_{\iota_b})x_m) \\ &= \left(\begin{pmatrix} \bar{x}'_1 \\ \vdots \\ \bar{x}'_a \end{pmatrix} \otimes \iota_b \right) - \iota_n \bar{x}', \end{aligned}$$

where the third equality follows from applying the argument in (5) to each of $((M_{\iota_a} \otimes P_{\iota_b})x_i)_{i=1}^m$. Similarly,

$$\begin{aligned} (P_{\iota_a} \otimes M_{\iota_b})X &= (P_{\iota_a} \otimes M_{\iota_b})(x_1, \dots, x_m) = ((P_{\iota_a} \otimes M_{\iota_b})x_1, \dots, (P_{\iota_a} \otimes M_{\iota_b})x_m) \\ &= \left(\iota_a \otimes \begin{pmatrix} \bar{x}'_1 \\ \vdots \\ \bar{x}'_a \end{pmatrix} \right) - \iota_n \bar{x}', \end{aligned}$$

where the third equality follows from applying the argument in (6) to each of $((P_{\iota_a} \otimes M_{\iota_b})x_i)_{i=1}^m$. Thus,

$$(I - P)X = X - \left(\begin{pmatrix} \bar{x}'_1 \\ \vdots \\ \bar{x}'_a \end{pmatrix} \otimes \iota_b \right) - \left(\iota_a \otimes \begin{pmatrix} \bar{x}'_1 \\ \vdots \\ \bar{x}'_b \end{pmatrix} \right) + \iota_n \bar{x}'.$$

The OLS estimates of π , μ , and ν are given by

$$\begin{aligned} \hat{\pi} &= \bar{y} - \bar{x}'\hat{\beta} \\ \hat{\mu}_i &= (\bar{y}_i - \bar{y}) - (\bar{x}_i - \bar{x})'\hat{\beta} \\ \hat{\nu}_t &= (\bar{y}_t - \bar{y}) - (\bar{x}_t - \bar{x})'\hat{\beta}. \end{aligned}$$

To see this, note that the OLS estimates of π , μ^* , and ν^* can be obtained from the regression

$$y - X\hat{\beta} = \iota_n \pi + (H_a \otimes \iota_b)\mu^* + (\iota_a \otimes H_b)\nu^* + \varepsilon.$$

Since the regressors are orthogonal, we may run three separate regressions. For example,

$$\hat{\pi} = (\iota_n' \iota_n)^{-1} \iota_n' (y - X\hat{\beta}) = \bar{y} - \bar{x}'\hat{\beta},$$

and

$$\begin{aligned}
\hat{\mu}^* &= ((H_a \otimes \iota_b)'(H_a \otimes \iota_b))^{-1} (H_a \otimes \iota_b)'(y - X\hat{\beta}) \\
&= \left((H_a' H_a)^{-1} \otimes \frac{1}{b} \right) (H_a \otimes \iota_b)'(y - X\hat{\beta}) \\
&= (H_a \otimes \frac{1}{b} \iota_b)'(y - X\hat{\beta}).
\end{aligned}$$

Here, we have

$$\hat{\mu} = H_a \hat{\mu}^* = (H_a H_a' \otimes \frac{1}{b} \iota_b)'(y - X\hat{\beta})$$

where

$$\begin{aligned}
(H_a H_a' \otimes \frac{1}{b} \iota_b)'(y - X\hat{\beta}) &= ((I_a - P_{\iota_a}) \otimes \frac{1}{b} \iota_b)'(y - X\hat{\beta}) \\
&= ((I_a - P_{\iota_a}) \otimes \frac{1}{b} \iota_b)'y - ((I_a - P_{\iota_a}) \otimes \frac{1}{b} \iota_b)'X\hat{\beta}
\end{aligned}$$

The first term on the right hand side can be written as

$$\begin{aligned}
((I_a - P_{\iota_a}) \otimes \frac{1}{b} \iota_b)'y &= \text{vec} \left((I_a - P_{\iota_a}) Y \frac{1}{b} \iota_b \right) \\
&= \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_a \end{pmatrix} - \iota_a \bar{y}.
\end{aligned} \tag{7}$$

For the second term on the right hand side, note that we have

$$\begin{aligned}
((I_a - P_{\iota_a}) \otimes \frac{1}{b} \iota_b)'X\hat{\beta} &= \sum_{i=1}^m ((I_a - P_{\iota_a}) \otimes \frac{1}{b} \iota_b)'x_i \hat{\beta}_i \\
&= \left(\begin{pmatrix} \bar{x}'_1 \\ \vdots \\ \bar{x}'_a \end{pmatrix} - \iota_a \bar{x}' \right) \hat{\beta},
\end{aligned}$$

where we've used (7) applied to each of $(x_i)_{i=1}^m$ in place of y . We may similarly get $\hat{\nu}$.

2 Random Effects Model

We consider the random effects panel model given by

$$y_{it} = \pi + x'_{it}\beta + \mu_i + \nu_t + \varepsilon_{it}$$

for $i = 1, \dots, a$ and $t = 1, \dots, b$ where $x_{it}, \in \mathbb{R}^m$ are not random and $(\mu_i), (\nu_t), (\varepsilon_{it})$ are mutually independent random variables with variances given by $\sigma_\mu^2, \sigma_\nu^2, \sigma_\varepsilon^2$. We may write the model in vector form as

$$y = \iota_n \pi + X\beta + u, \quad (8)$$

where

$$u = (I_a \otimes \iota_b)\mu + (\iota_a \otimes I_b)\nu + \varepsilon.$$

The variance Σ of u is given by

$$\Sigma = \sigma_\mu^2(I_a \otimes \iota_b \iota_b') + \sigma_\nu^2(\iota_a \iota_a' \otimes I_b) + \sigma_\varepsilon^2 I_n.$$

To analyze the GLS estimates of β and π , we define the following orthogonal projections that are mutually orthogonal and sum up to I :

$$\begin{aligned} P_1 &= \left(I_a - \frac{\iota_a \iota_a'}{a} \right) \otimes \left(I_b - \frac{\iota_b \iota_b'}{b} \right) \\ P_2 &= \left(I_a - \frac{\iota_a \iota_a'}{a} \right) \otimes \frac{\iota_b \iota_b'}{b} \\ P_3 &= \frac{\iota_a \iota_a'}{a} \otimes \left(I_b - \frac{\iota_b \iota_b'}{b} \right) \\ P_4 &= \frac{\iota_a \iota_a'}{a} \otimes \frac{\iota_b \iota_b'}{b}. \end{aligned}$$

Note that we have

$$\begin{aligned} I_n &= (I_a \otimes I_b) = ((P_{\iota_a} + M_{\iota_a}) \otimes (P_{\iota_b} + M_{\iota_b})) \\ &= (P_{\iota_a} \otimes P_{\iota_b}) + (P_{\iota_a} \otimes M_{\iota_b}) + (M_{\iota_a} \otimes P_{\iota_b}) + (M_{\iota_a} \otimes M_{\iota_b}) \\ &= P_4 + P_3 + P_2 + P_1. \end{aligned}$$

We may rewrite Σ as

$$\begin{aligned} \Sigma &= \sigma_\mu^2(I_a \otimes \iota_b \iota_b') + \sigma_\nu^2(\iota_a \iota_a' \otimes I_b) + \sigma_\varepsilon^2 I_n \\ &= \sigma_\mu^2 b(P_2 + P_4) + \sigma_\nu^2 a(P_3 + P_4) + \sigma_\varepsilon^2(P_1 + P_2 + P_3 + P_4) \\ &= \sigma_\varepsilon^2 P_1 + (b\sigma_\mu^2 + \sigma_\varepsilon^2) P_2 + (a\sigma_\nu^2 + \sigma_\varepsilon^2) P_3 + (b\sigma_\mu^2 + a\sigma_\nu^2 + \sigma_\varepsilon^2) P_4 \\ &= \sum_{i=1}^4 \lambda_i P_i \end{aligned}$$

Since $(P_i)_{i=1}^4$ are orthogonal projections on orthogonal spaces, they form eigenspaces of Σ with eigenvalues given by (λ_i) .² The GLS transformation of the model (8) is given by

$$\Sigma^{-1/2} y = \Sigma^{-1/2} \iota_n \pi + \Sigma^{-1/2} X\beta + \Sigma^{-1/2} u, \quad (9)$$

²Note that (λ_i) are defined differently here, compared to lecture notes.

where $\text{var}(\Sigma^{-1/2}u) = I_n$ by construction. Here, note that

$$\Sigma^{-1/2}\iota_n = \sum_{i=1}^4 \lambda_i^{-1/2} P_i \iota_n = \lambda_4^{-1/2} \iota_n$$

because $\iota_n = (\iota_a \otimes \iota_b)$ is orthogonal to $P_i, i = 1, 2, 3$. Thus, by applying FWL to the model (9), we have

$$\hat{\beta}_{gls} = (X' \Sigma^{-1/2} M_{\iota_n} \Sigma^{-1/2} X)^{-1} X' \Sigma^{-1/2} M_{\iota_n} \Sigma^{-1/2} y.$$

Also, $M_{\iota_n} = I_n - (P_{\iota_a} \otimes P_{\iota_b}) = I_n - P_4$, so that

$$\begin{aligned} \hat{\beta}_{gls} &= (X' \Sigma^{-1/2} M_{\iota_n} \Sigma^{-1/2} X)^{-1} X' \Sigma^{-1/2} M_{\iota_n} \Sigma^{-1/2} y \\ &= \left(X' \left(\sum_{i=1}^3 \lambda_i^{-1} P_i \right) X \right)^{-1} X' \left(\sum_{i=1}^3 \lambda_i^{-1} P_i \right) y \\ &= \left(X' \left(\frac{1}{\sigma_\varepsilon^2} P_1 + \frac{1}{b\sigma_\mu^2 + \sigma_\varepsilon^2} P_2 + \frac{1}{a\sigma_\nu^2 + \sigma_\varepsilon^2} P_3 \right) X \right)^{-1} X' \left(\frac{1}{\sigma_\varepsilon^2} P_1 + \frac{1}{b\sigma_\mu^2 + \sigma_\varepsilon^2} P_2 + \frac{1}{a\sigma_\nu^2 + \sigma_\varepsilon^2} P_3 \right) y \\ &= \left(X' \left(P_1 + \frac{\sigma_\varepsilon^2}{b\sigma_\mu^2 + \sigma_\varepsilon^2} P_2 + \frac{\sigma_\varepsilon^2}{a\sigma_\nu^2 + \sigma_\varepsilon^2} P_3 \right) X \right)^{-1} X' \left(P_1 + \frac{\sigma_\varepsilon^2}{b\sigma_\mu^2 + \sigma_\varepsilon^2} P_2 + \frac{\sigma_\varepsilon^2}{a\sigma_\nu^2 + \sigma_\varepsilon^2} P_3 \right) y. \end{aligned}$$

Here, the projection matrix is approximately

$$\begin{aligned} P_1 + \frac{\sigma_\varepsilon^2}{b\sigma_\mu^2 + \sigma_\varepsilon^2} P_2 + \frac{\sigma_\varepsilon^2}{a\sigma_\nu^2 + \sigma_\varepsilon^2} P_3 &\approx P_1 = I_n - P_4 - P_2 - P_3 \\ &= I_n - (P_{\iota_a} \otimes P_{\iota_b}) - (M_{\iota_a} \otimes P_{\iota_b}) - (P_{\iota_a} \otimes M_{\iota_b}). \end{aligned}$$

for a, b large. This implies that fixed effects and random effects estimators are similar when the sample size is large. The GLS estimate of π can be obtained from the OLS in

$$\Sigma^{-1/2}y - \Sigma^{-1/2}X\hat{\beta}_{gls} = \Sigma^{-1/2}\iota_n\pi + \Sigma^{-1/2}u$$

so that

$$\begin{aligned} \hat{\pi} &= (\iota_n' \Sigma^{-1} \iota_n)^{-1} \iota_n' \Sigma^{-1} (y - X\hat{\beta}) \\ &= \left(\frac{1}{\lambda_4} \right)^{-1} \left(\frac{1}{\lambda_4} P_4 \right) (y - X\hat{\beta}) \\ &= P_4 (y - X\hat{\beta}) \\ &= \bar{y} - \bar{x}' \hat{\beta}. \end{aligned}$$