

Review Session 5

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1 Proofs of Fundamental Lemmas

1.1 Key Idea

Consider the partial sum process

$$B_n(r) = \frac{1}{\sqrt{n}} \sum_{t=1}^{\lfloor nr \rfloor} w_t.$$

We have

$$B_n(r) = \begin{cases} 0 & r \in [0, \frac{1}{n}) \\ \frac{1}{\sqrt{n}} w_1 & r \in [\frac{1}{n}, \frac{2}{n}) \\ \frac{1}{\sqrt{n}} (w_1 + w_2) & r \in [\frac{2}{n}, \frac{3}{n}) \\ \vdots & \\ \frac{1}{\sqrt{n}} (w_1 + \dots + w_{n-1}) & r \in [\frac{n-1}{n}, 1) \\ \frac{1}{\sqrt{n}} (w_1 + \dots + w_n) & r = 1 \end{cases}$$

Using the definition of the Riemann integral, we may write

$$\int_0^1 B_n(r) dr = \sum_{k=1}^n B_n\left(\frac{k-1}{n}\right) \frac{1}{n},$$

where the equality is exact without taking the limit because $B_n(r)$ is a step function. The invariance principle tells us that

$$B_n(r) \rightarrow_d B(r)$$

for each $r \in [0, 1]^1$ where $B(r)$ is a Brownian motion with variance given by

$$\begin{aligned}
\text{Var}(B(r)) &= \mathbb{E}B(r)B(r)' = \lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left[\left(\sum_{t=1}^{[nr]} w_t \right) \left(\sum_{t=1}^{[nr]} w_t \right)' \right] \\
&= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=-([nr]-1)}^{[nr]-1} ([nr] - |k|) \Gamma(k) \\
&= r \lim_{n \rightarrow \infty} \sum_{k=-(m-1)}^{m-1} \left(\frac{[nr]}{nr} - \frac{|k|}{nr} \right) \Gamma(k) \\
&= r \lim_{m \rightarrow \infty} \sum_{k=-(m-1)}^{m-1} \left(1 - \frac{|k|}{m} \right) \Gamma(k) \\
&= r \sum_{k=-\infty}^{\infty} \Gamma(k)
\end{aligned}$$

where we took $m = [nr]$ and used $\frac{[nr]}{nr} \rightarrow 1$ as $m \rightarrow \infty$. Since the mapping $B \mapsto \int_0^1 B(r)dr$ is continuous, using the continuous mapping theorem, we have

$$\int_0^1 B_n(r)dr \rightarrow_d \int_0^1 B(r)dr$$

In other words,

$$\sum_{k=1}^n B_n \left(\frac{k-1}{n} \right) \frac{1}{n} = \sum_{k=1}^n \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{k-1} w_i \right) \frac{1}{n} \rightarrow_d \int_0^1 B(r)dr$$

By defining $z_t = \sum_{s=1}^t w_s$, we may write

$$\frac{1}{n^{3/2}} \sum_{t=1}^{n-1} z_t \rightarrow_d \int_0^1 B(r)dr.$$

Together with

$$\frac{1}{n^{3/2}} z_n = \frac{1}{n} \left(\frac{1}{\sqrt{n}} \sum_{t=1}^n w_t \right) = O_p \left(\frac{1}{n} \right)$$

we have

$$\frac{1}{n^{3/2}} \sum_{t=1}^n z_t \rightarrow_d \int_0^1 B(r)dr$$

¹To be more precise, the convergence in distribution $B_n \rightarrow_d B$ holds under the Skorokhod J_1 topology.

1.2 Four Fundamental Lemmas

Let

$$z_t = \sum_{s=1}^t w_s, \quad z_0 = 0, \quad \Gamma(k) = \mathbb{E}[w_t w'_{t-k}], \quad \Sigma = \Gamma(0), \quad \Lambda = \sum_{k=1}^{\infty} \Gamma(k), \quad \Omega = \Sigma + \Lambda + \Lambda'.$$

We first state the four fundamental lemmas. In stating and using the fundamental lemmas, we should always keep in mind: By what order do we need to divide?

1.

$$\sum_{t=1}^n c_{it} z_t \implies \sum_{t=1}^n \frac{c_{it}}{n^{\delta_i}} \frac{z_t}{\sqrt{n}} \frac{1}{n} \rightarrow_d \int_0^1 f_i(r) B(r) dr$$

2.

$$\begin{aligned} \sum_{t=1}^n c_{it} w_t &\implies \sum_{t=1}^n \frac{c_{it}}{n^{\delta_i}} \frac{w_t}{\sqrt{n}} = \sum_{t=2}^n \frac{c_{it}}{n^{\delta_i}} \left(\frac{z_t}{\sqrt{n}} - \frac{z_{t-1}}{\sqrt{n}} \right) + \frac{c_{i1}}{n^{\delta_i}} \frac{z_1}{\sqrt{n}} \\ &\rightarrow_d \int_0^1 f_i(r) dB(r) \end{aligned}$$

3.

$$\begin{aligned} \sum_{t=1}^n z_t z'_t &\implies \sum_{t=1}^n \frac{z_t}{\sqrt{n}} \frac{z'_t}{\sqrt{n}} \frac{1}{n} \\ &\rightarrow_d \int_0^1 B(r) B(r)' dr \end{aligned}$$

4.

$$\begin{aligned} \sum_{t=2}^n z_{t-1} w'_t &\implies \sum_{t=2}^n \frac{z_{t-1}}{\sqrt{n}} \frac{w'_t}{\sqrt{n}} = \sum_{t=2}^n \frac{z_{t-1}}{\sqrt{n}} \left(\frac{z_t}{\sqrt{n}} - \frac{z_{t-1}}{\sqrt{n}} \right)' \\ &\rightarrow_d \int_0^1 B(r) dB(r)' + \Lambda' \end{aligned}$$

Proof. The Lemmas 1 and 3 follow from the definition of the Riemann integral and the continuous mapping theorem. Notice that we may write

$$\begin{aligned} \sum_{t=1}^n \frac{c_{it}}{n^{\delta_i}} \frac{z_t}{\sqrt{n}} \frac{1}{n} &= \sum_{t=1}^{n-1} \frac{c_{it}}{n^{\delta_i}} \frac{z_t}{\sqrt{n}} \frac{1}{n} + \frac{c_{in}}{n^{\delta_i}} \frac{z_n}{\sqrt{n}} \frac{1}{n} = \int_0^1 f_{ni}(r) B_n(r) dr + \frac{c_{in}}{n^{\delta_i}} \frac{z_n}{\sqrt{n}} \frac{1}{n} \\ \sum_{t=1}^n \frac{z_t}{\sqrt{n}} \frac{z'_t}{\sqrt{n}} \frac{1}{n} &= \sum_{t=1}^{n-1} \frac{z_t}{\sqrt{n}} \frac{z'_t}{\sqrt{n}} \frac{1}{n} + \frac{z_n}{\sqrt{n}} \frac{z'_n}{\sqrt{n}} \frac{1}{n} = \int_0^1 B_n(r) B_n(r)' dr + \frac{z_n}{\sqrt{n}} \frac{z'_n}{\sqrt{n}} \frac{1}{n} \end{aligned}$$

where both identities are exact due to $B_n(r)$ and $f_{ni}(r) = \frac{c_{i[nr]}}{n^{\delta_i}}$ being step functions. Note that the second terms in each equality are asymptotically negligible, and

$$\begin{aligned} B &\mapsto \int_0^1 f_i(r) B(r) dr \\ B &\mapsto \int_0^1 B(r) B(r)' dr \end{aligned}$$

are continuous so that we may use the invariance principle and the continuous mapping theorem to finish. Lemma 2 also follows from IP and CMT. We first apply summation by parts

$$\begin{aligned}
\sum_{t=1}^n \frac{c_{it}}{n^{\delta_i}} \frac{w_t}{\sqrt{n}} &= \sum_{t=2}^n \frac{c_{it}}{n^{\delta_i}} \left(\frac{z_t}{\sqrt{n}} - \frac{z_{t-1}}{\sqrt{n}} \right) + \frac{c_{i1}}{n^{\delta_i}} \frac{z_1}{\sqrt{n}} \\
&= \sum_{t=1}^n \frac{c_{it}}{n^{\delta_i}} \frac{z_t}{\sqrt{n}} - \sum_{t=2}^n \frac{c_{it}}{n^{\delta_i}} \frac{z_{t-1}}{\sqrt{n}} \\
&= \frac{c_{in}}{n^{\delta_i}} \frac{z_n}{\sqrt{n}} + \sum_{t=1}^{n-1} \frac{c_{it}}{n^{\delta_i}} \frac{z_t}{\sqrt{n}} - \sum_{t=2}^n \frac{c_{it}}{n^{\delta_i}} \frac{z_{t-1}}{\sqrt{n}} \\
&= \frac{c_{in}}{n^{\delta_i}} \frac{z_n}{\sqrt{n}} - \sum_{t=2}^n \frac{z_{t-1}}{\sqrt{n}} \left(\frac{c_{it} - c_{it-1}}{n^{\delta_i}} \right) \\
&= f_{ni}(1)B_n(1) - \int_0^1 B_n(r) df_{ni}(r)
\end{aligned}$$

where the final equality is exact by the definition of the Riemann-Stieltjes integral and because $B_n(r)$ and $f_{ni}(r) = \frac{c_{i[nr]}}{n^{\delta_i}}$ are step functions. Now since the map $B \mapsto f_i(1)B(1) - \int_0^1 B(r) df_i(r)$ is continuous, we use IP and CMT to get

$$f_{ni}(1)B_n(1) - \int_0^1 B_n(r) df_{ni}(r) \rightarrow_d f_i(1)B(1) - \int_0^1 B(r) df_i(r)$$

Finally, using integration by parts, we have²

$$\int_0^1 f_i(r) dB(r) = f_i(1)B(1) - f_i(0)B(0) - \int_0^1 B(r) df_i(r)$$

so that we have

$$\sum_{t=1}^n \frac{c_{it}}{n^{\delta_i}} \frac{w_t}{\sqrt{n}} \rightarrow_d \int_0^1 f_i(r) dB(r)$$

For the proof of Lemma 4, we may not simply apply IP and CMT because the map $B \mapsto \int_0^1 B(r) dB(r)$ is not continuous. This is why Λ' appears in Lemma 4. Consider the scalar version of Lemma 4. We see that

$$\frac{1}{n} \sum_{t=2}^n z_{t-1} w_t = \frac{1}{2} \left(\left(\frac{1}{\sqrt{n}} \sum_{t=1}^n w_t \right)^2 - \frac{1}{n} \sum_{t=1}^n w_t^2 \right)$$

because $\frac{1}{n} \sum_{t=2}^n z_{t-1} w_t$ collects the cross product terms in $\left(\frac{1}{\sqrt{n}} \sum_{t=1}^n w_t \right)^2$. Now, we see that

$$\begin{aligned}
\frac{1}{2} \left(\left(\frac{1}{\sqrt{n}} \sum_{t=1}^n w_t \right)^2 - \frac{1}{n} \sum_{t=1}^n w_t^2 \right) &= \frac{1}{2} \left(B_n(1)^2 - \frac{1}{n} \sum_{t=1}^n w_t^2 \right) \\
&\rightarrow_d \frac{1}{2} (B(1)^2 - \sigma^2)
\end{aligned}$$

²Note that the quadratic covariation term, i.e., $[f_i, B](r)$ is zero because f_i is of bounded variation.

due to IP and CMT. Also, using Ito formula on $f(x) = x^2$, we have

$$d(B(t)^2) = 2B(t)dB(t) + \frac{1}{2}2d[B](t)$$

integrating over $[0, 1]$, we have

$$B(1)^2 = 2 \int_0^1 B(t)dB(t) + w^2$$

because the quadratic variation of B is given by $[B](t) = w^2 t$ where $w^2 = \sum_{k=-\infty}^{\infty} \gamma(k)$ is the long run variance of (w_t) . Thus, we have

$$\frac{1}{n} \sum_{t=2}^n z_{t-1} w_t \rightarrow_d \int_0^1 B(t)dB(t) + \frac{w^2 - \sigma^2}{2}$$

□

Remarks

1. For an intuitive understanding of why the term $\frac{w^2 - \sigma^2}{2} = \sum_{k=1}^{\infty} \gamma(k)$ appears on Lemma 4., note that in using the IP,

$$B_n(r) = \frac{1}{\sqrt{n}} \sum_{t=1}^{[nr]} w_t \rightarrow_d B(r)$$

the limit $B(r)$, by virtue of being a Brownian motion, has the independent increment property, i.e.,

$$B(s) \perp (B(t) - B(s))$$

for all $s < t$. However, we allow serial dependencies when using IP so that the partial sum process $B_n(r)$ generally does not have independent increments. In other words,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^s w_i \perp \frac{1}{\sqrt{n}} \sum_{i=s+1}^t w_i$$

does not hold in general. Suppose we have

$$B_n(r) = \begin{pmatrix} B_{n1}(r) \\ B_{n2}(r) \end{pmatrix} \rightarrow_d B(r) = \begin{pmatrix} B_1(r) \\ B_2(r) \end{pmatrix}$$

and consider the Ito integral given by

$$\int_0^1 B_2(r)dB_1(r)' := \lim_{\pi_n \rightarrow 0} \sum_{i=1}^n B_2(t_{i-1}) (B_1(t_i) - B_1(t_{i-1}))'$$

for partition $0 = t_0 < t_1 < \dots < t_n = 1$ of $[0, 1]$ with mesh size $\pi_n = \max_i |t_i - t_{i-1}|$. Since $B(r)$ has independent increments, we have

$$\begin{aligned} \mathbb{E} \left[\sum_{i=1}^n B_2(t_{i-1}) (B_1(t_i) - B_1(t_{i-1}))' \right] &= \sum_{i=1}^n \mathbb{E} [B_2(t_{i-1}) (B_1(t_i) - B_1(t_{i-1}))'] \\ &= 0 \end{aligned}$$

for any choice of partition (t_i) , implying that we have

$$\mathbb{E} \left[\int_0^1 B_2(r) dB_1(r)' \right] = 0$$

However, because $B_n(r)$ need not have independent increments, we do not have

$$\mathbb{E} \left[\sum_{t=1}^n B_{2n} \left(\frac{t-1}{n} \right) \left(B_{1n} \left(\frac{t}{n} \right) - B_{1n} \left(\frac{t-1}{n} \right) \right)' \right] = 0$$

In fact,

$$\begin{aligned} \mathbb{E} \left[\sum_{t=1}^n B_{2n} \left(\frac{t-1}{n} \right) \left(B_{1n} \left(\frac{t}{n} \right) - B_{1n} \left(\frac{t-1}{n} \right) \right)' \right] &= \sum_{t=1}^n \mathbb{E} \left[B_{2n} \left(\frac{t-1}{n} \right) \left(B_{1n} \left(\frac{t}{n} \right) - B_{1n} \left(\frac{t-1}{n} \right) \right)' \right] \\ &= \sum_{t=1}^n \mathbb{E} \left[\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{t-1} w_{2k} \right) \left(\frac{1}{\sqrt{n}} w_{1t} \right)' \right] \\ &= \frac{1}{n} \sum_{t=1}^n \sum_{k=1}^{t-1} \mathbb{E} [w_{2k} w_{1t}'] \\ &= \frac{1}{n} \sum_{k=1}^{n-1} (n-k) \mathbb{E} [w_{2t-k} w_{1t}'] \\ &= \sum_{k=1}^{n-1} \left(1 - \frac{k}{n} \right) \Gamma_{12}(k)' \\ &\rightarrow \sum_{k=1}^{\infty} \Gamma_{12}(k)' \end{aligned}$$

where

$$\Gamma(k) = \mathbb{E} w_t w_{t-k}' = \begin{pmatrix} \Gamma_{11}(k) & \Gamma_{12}(k) \\ \Gamma_{21}(k) & \Gamma_{22}(k) \end{pmatrix}$$

2. Since we have

$$\frac{1}{n} \sum_{t=1}^n z_t w_t' = \frac{1}{n} \sum_{t=2}^n z_{t-1} w_t' + \frac{1}{n} \sum_{t=1}^n w_t w_t'$$

using Lemma 4, we have

$$\begin{aligned} \frac{1}{n} \sum_{t=1}^n z_t w_t' &= \frac{1}{n} \sum_{t=2}^n z_{t-1} w_t' + \frac{1}{n} \sum_{t=1}^n w_t w_t' \\ &\rightarrow_d \int_0^1 B(t) dB(t)' + \Lambda' + \Sigma \end{aligned}$$

where

$$\Lambda = \sum_{k=1}^{\infty} \Gamma(k), \quad \Sigma = \Gamma(0)$$